$$|[EF]=[|E|F],$$

where $[e_1e_2\ldots\ldots e_n]=1$.

The regressive products can be made plainer by an example. Let 5 be the order of the space considered, $[e_1e_2\ldots\ldots e_5]=1$, and let the product of $E_1=[e_1e_2e_3e_4]$ and $E_2=[e_1e_2e_5]$ be required.

Changing the order of the factors of E_2, we write $E_2=[e_5e_1e_2]$ (37), there being two interchanges. Then

$$[E_1E_2]=[e_1e_2e_3e_4][e_5e_1e_2]=[e_1e_2e_3e_4e_5e_1e_2]\ (52)=[e_1e_2]\ \text{(Rem. Art. 13)}.$$

Thus the product of E_1 and E_2 is the product of their common factors e_1 and e_2. The question arises, how can the common factors be selected and their product formed out of the two given factors?

Since E_1 and E_2 together contain all five of the units and neiter $|E_1$ nor $|E_2$ contains e_1 or e_2, the product of $|E_1$ and $|E_2$ contains all the units except e_1 and e_2. Then $|[E_1E_2]$ contains the factors of $[|E_1|E_2]$. Hence the definition of a regressive product.

62. *If q and r are the orders of A and B and n that of the space considered, the order of the product $[AB]$ is equal to $q+r$ when $q+r<n$, but is equal to $q+r-n$ when $q+r\geqq n$. In the language of the Theory of Numbers if p is the order of the product*

$$p\equiv q+r\ (\text{Modulus } n).$$

The proof of this theorem follows the lines of the example of the preceding article.

63. *Similarly, for a larger number of factors,*

$$p\equiv q+r+s+t\ldots\ldots\ (\text{Modulus } n).$$

64. *The product of the complements of two quantities is the complement of the product of those quantities, that is to say,*

$$[|A\,|B]=|[AB].$$

Proof.—1. Suppose at first that the sum of the orders α and β of A and B is greater that n, that of the space under consideration. Let $A=\Sigma\alpha_rE_r$, $B=\Sigma\beta_sF_s$, where E_r and F_s are units. Then $|A=\Sigma\alpha_r|E_r$ and $|B=\Sigma\beta_s|F_s$ (58). Thus, we have,

$$[|A\,|B]\equiv[\Sigma\alpha_r|E_r.\Sigma\beta_s|F_s]=\Sigma\alpha_r\beta_s[|E_r|F_s\ \ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\alpha_r\beta_s|[E_rF_s]\ \ \ldots\ldots(61)=|\Sigma\alpha_r\beta_s[E_rF_s]\ \ldots\ldots(58)$$

$$=|[\Sigma\alpha_rE_r.\Sigma\beta_sF_s]\ldots\ldots(28)\equiv|[AB].$$

2. Suppose $\alpha+\beta=n$. Let E and F be products of the original units. Two cases may be distinguished. First, when E and F contain a common factor e_1. It is plain that in this case both $[EF]$ and $[\,|E|F]$ contain common factors, so that they are each equal to zero (43). Second, when $[EF]=1$. Replacing $|E$ and $|F$ by their values from 57 and noting that $[FE][FE]=+1$, we get $[\,|E\,|F]=[EF]$. But as $[EF]=1$, $|[EF]=1$ (57). Thus the law holds for units. Then reasoning as in 1, above, it holds for any quantities.

3. Suppose $\alpha+\beta<n$. The proof of this case is based on 1. of this article by letting $A=|A'$, $B=|B'$, and writing

$$|\,[A'B']=[|A'\,|B']=[AB].$$

65. *The product of the complements of several quantities is the complement of the product of these quantities.*

Proved by mathematical induction from 64.

66. *The complement of a polynomial is the sum of the complements of its parts.*

Proof from 58.

67. *If E, F, G are units the sum of whose orders is n (the order of the space considered),*

$$[EF.EG]=[EFG]E.$$

Proof.—We distinguish two cases: either $[EFG]$ contains equal factors or it does not. If it does, then at least one of the units, say e_1, is missing in it. Now let $[EF]=|Q$; then by 57 Q must contain e_1; likewise, let $[EG]=|R$; then R contains e_1. Then $[QR]=0$ (43). Now we have

$$[EF.EG]=[\,|Q\,|R]=|\,[QR]\ (64)=|0=0\ (57).$$

But $[EFG]$ also equals zero since it contains equal factors (43). Thus in this case

$$[EF.EG]=[EFG]E.$$

If $[EFG]$ does not contain equal factors, then it contains all of the original units and no others. Then by 57 and 55,

$$|G=[GEF][EF],\quad |F=[FEG][EG].$$

Since $[GEF]$ and $[FEG]$ are equal to either $+1$ or -1, they can appear on either side of their equations. Hence we can write

$$[EF]=[GEF]\,|G,\quad [EG]=[FEG]\,|F.\quad \text{Whence}$$

$$[EF.EG]=[GEF][FEG][\,|G\,|F]=[GEF][FEG]\,|\,[GF\ldots\ldots\ldots(64)$$

$$=[GEF][FEG][GFE]E....(57)=[GEF][EFG][GEF]E....(41)$$

$$=[EFG]E,$$

since $[GEF][GEF]$, as in (57), equals $+1$.

68. *If A, B, C are simple quantities the sum of whose orders equals n, the order of the space of these quantities.*

$$[AB.AC]=[ABC]A.$$

The proof of this formula (based on (67) on account of its length, is omitted, as are also those of the next four formulas given below.

69.—71. *If A, B, C are simple quantities whose product is of the 0th order.*

69. $$[AB.AC]=[ABC]A.$$

70. $$[AB.BC]=[ABC]B.$$

71. $$[AC.BC]=[ABC]C.$$

72. *If A, B, C are simple quantities and the sum of the orders of A and C equals the order of the space considered and B is subordinate* (18) *to A, then*

$$[A.BC]=[AC]B \text{ and } [CB.A]=[CA]B.$$

Remark.—It seems proper to state here that the matter contained in Chapters II—V is taken direct from Grassmann's Ausdehnungslehre of 1862. What the writer has done has been to cut out everything which was not essential to the development of the main principles of the work. What to insert and what to omit constitutes the chief difficulty. In the following chapters (except Chapter VIII) we shall not follow Grassmann very closely.

MATHEMATICAL INDUCTION.

By ARTHUR L. BAKER, C. E., Ph. D., University of Rochester, Rochester, N. Y.

There is such a general lack of presentation of the governing principles of mathematical induction in the text books which refer to the subject, and failure to give a working rule for its application, that it is thought a presentation of such a working rule specifically stated and not left to inference from examples merely, would be acceptable.

Mathematical Induction (not entering into the question of the appropriateness of the name) is a method of proof used where a primary operation and a

secondary law have produced the same series of results to see if the supposed secondary law is valid.

It may be conveniently divided into two cases :

I. When the primary operation is an algebraic one and the secondary law a functional one.

II. When the primary oporation is a functional one and the secondary law an algebraic one.

The necessity for a proof that the secondary law is valid is seen when we find that many apparent secondary laws which hold for a few terms fail later on, in other words, that because an apparent law holds for ten, twenty or thirty terms of a series, we are not at all justified in inferring that it will hold throughout.

A few examples will illustrate this :

x^2+x+17 is prime for sixteen values of x, beginning with $x=0$, but the law fails for the seventeenth term of the series.

x^2+x+41 is likewise prime for forty terms, but the law fails on the forty-first.

The sum of the divisors of n ! is $\frac{1}{2}(n+1)!$ for a few terms, after which the law fails. And so on for many apparent secondary laws.

I. When the primary operation is a simple algebraic one, and the supposed secondary law is a more complex or functional one.

In this case Mathematical Induction consists in applying the primary operation to the general result produced by the secondary law to see if the next term of the series thus produced by the primary operation is the same as would result from an application of the secondary law. If it is, and the operand was correct, the secondary law is correct.

Example.—$1^3+2^3=9(2-1)^2$, $1^3+2^3+3^3=9(3-1)^2$, giving the apparent secondary law $1^3+2^3+\ \ldots\ . n^3=9(n-1)^2$.

To test this, add another term (primary operation) giving

$$1^3+2^3+\ldots\ldots(n+1)^3=9(n-1)^2+(n+1)^3\pm 9n^2,$$

and therefore the secondary law is not correct.

Example.—$1^3+2^3=\left(2\frac{2+1}{2}\right)^2$, $1^3+2^3+3^3=\left(3\frac{3+1}{2}\right)^2$, giving the apparent secondary law,

$$1^3+2^3+\ldots\ldots n^3=\left(n\frac{n+1}{2}\right)^2.$$

Apply the primary operation by adding another term, giving

$$1^3+2^3+\ldots\ldots(n+1)^3=\left(n\frac{n+1}{2}\right)^2+(n+1)^3=\left[(n+1)\ \frac{n+2}{2}\right]^2$$

and the secondary law is correct.

II. When the primary operation is a functional one and the secondary law is a simple algebraic one.

In this case mathematical induction consists in stating the primary functional operation for two general terms in succeession, and taking any convenient multiple of the first from a convenient multiple of the second. If this difference conforms to the supposed secondary law, the secondary law is true provided the first functional result is true.

Thus let

$$f(x)=F\phi$$

be a term of the series, where x is the element whose change produces the term of the series, and ϕ is the expression which enters as an element of the secondary law. If now

$$A.f(x+1)-B.f(x)=F_1\phi$$

where F_1 has the same algebraic form as F, except as to the values of the constants, then

$$f(x+1)=F_2\phi$$

where F_2 has the same algebraic form as F and F_1 except as to the value of the constants. Hence the form of the function F being persistent for two terms, the law is true if true for F.

A special case is where $F\phi=C.\phi$, C being a mere factor.

Example. — $x^3-x=\begin{pmatrix}6\alpha\\24\beta\end{pmatrix}$ according as x is $\begin{pmatrix}\text{even}\\\text{odd}\end{pmatrix}$, α and β being factors, holds for some cases.

1°. $x=2n+1$.

$f(x)=(2n+1)^3-(2n+1)=8n^3+12n^2+4n$.

$f(x+1)=(2n+3)^3-(2n+3)=8n^3+36n^2+52n+24$.

$f(x+1)-f(x)=24n^2+48n+24=24\beta$. Q. E. I.

2°. $x=2n$.

$f(x)=(2n)^3-2n=8n^3-2n$.

$f(x+1)=(2n+2)^3-(2n+2)=8n^3+24n^2+22n+6$.

$f(x+1)-f(x)=24n^2+24n+6=6\alpha$. Q. E. I.

Example.—$f(x)=5^{2x+2}-24x-25=576\alpha$ holds for $x=1, 2, 3$.

$f(x+1)=5^{2x+4}-24(x+1)-25$.

$f(x+1)-25f(x)=25(24x+25)-24x-49=576(x+1)=576\alpha$. Q. E. I.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

122. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Suppose 10% traction stock is 20% better in the market than 5% mining stock; if my income be $500 from each, how much money have I paid for each, the whole investment bringing $6\frac{2}{3}\%$?

Solution by BENJAMIN F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio; M. A. GRABER, Student in Heidelberg University, Tiffin, Ohio; E. L. SHERWOOD, Professor of Mathematics, Beaver College, Beaver, Pa.; and the PROPOSER.

$\$500 \div .10 = \$5000 =$ par value of traction stock;

$\$500 \div .05 = \$10000 =$ par value of mining stock;

$\$1000 \div .06\frac{2}{3} = \$15000 =$ whole investment.

$\$5000 = \frac{1}{2}$ of $\$10000$ or the face of the traction stock is one-half the face of the mining stock $= 50\%$.

$\therefore 100\% = 50\%$ or $1\% = \frac{1}{2}\%$.

$20\% = 20 \times \frac{1}{2}\% = 10\% =$ excess of traction stock over same amount of mining stock.

$50\% + 10\% = 60\% =$ investment in traction stock.

$100\% + 60\% = 160\% =$ whole investment.

$\$15000 \div 1.60 = \$9375 =$ amount invested in mining stock.

$\$15000 - \$9375 = \$5625 =$ amount invested in traction stock.

Also solved, with different results, by *ELMER SCHUYLER*, *ALOIS F. KOVARIK*, *COOPER D. SCHMITT*, and *LESLIE L. LOCKE*.

123. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

If $m = 2$ cents be the interest on $M = 100$ cents for $p = 40$ days, find the yearly rate per cent.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa; LESLIE L. LOCKE, Professor of Mathematics, Fredonia Institute, Fredonia, Pa.; ELMER SCHUYLER, Professor of German, Reading, Pa.; and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

If the interest on M cents for p days is m cents, the interest on M cents for 1 day is m/p cents, and for one year is $360m/p$ cents. Then the annual rate is $\frac{360m/p}{M}$ of 100% which equals $\frac{36000m}{pM}$. In the present example $m = 2$, $p = 40$, $M = 100$. Whence rate $= \frac{72000}{4000} = 18\%$.

ALGEBRA.

98. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A and B agreed to reap a field of grain for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days; but B, who did not work as quickly ae he expected, was obliged to call to his assistance C, an inferior workman, who worked the last two days, in consequence of which B received 3*s*. 6*d*. less than would otherwise have been due him. In what time could B and C each reap the field? From *Milne's High School Algebra*.

I. Solution by O. S. WESTCOTT, Principal of North Division High School, Chicago, Ill.

Let x=time in which B could reap the field.

y=time in which C could reap the field.

Then $1/x+\frac{1}{9}=\frac{x+9}{9x}$=what A and B can do in one day.

$\therefore \frac{9x}{x+9}$=time in which A and B could do the work.

Now $90/x$ represents B's daily wages.

Therefore, $\frac{90}{x}\left(\frac{9x}{x+9}\right)$=what B should have received if A and B had together completed the job.

But $90/x\times 5=\frac{450}{x}$ represents what B did receive.

Hence $\frac{90}{x}\left(\frac{9x}{x+9}\right)-\frac{450}{x}=3\frac{3}{4}$.

From which we obtain $x^2-87x=-1080$, and

$$x=\tfrac{87}{2}\pm\sqrt{(-1080+\tfrac{7569}{4})},\quad x=\tfrac{1}{2}(87\pm 57)=15 \text{ or } 72,$$

and since $\frac{5}{9}+5/x+2/y=1$, we have $\frac{5}{9}+\frac{5}{15}+2/y=1$ and $y=18$, or $\frac{5}{9}+\frac{5}{72}+2/y=1$ and $y=5\frac{1}{3}$.

But it is explicitly stated in the problem that C was the inferior workman; therefore the values $x=15$ and $y=18$ are the values sought.

Similar solutions were received from *COOPER D. SCHMITT, ALOIS F. KOVARIK, J. SCHEFFER, G. W. NASH*, and *J. M. BANDY*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In five days A does $\frac{5}{9}$ of the work and receives 50 shillings. B receives $40-3\frac{3}{4}=36\frac{1}{4}$ shillings for 5 days work, or $14\frac{1}{2}$ shillings for 2 days work.

$14\frac{1}{2}\div 3\frac{3}{4}=\frac{29}{2}\times\frac{4}{15}=\frac{58}{15}$ times as many days for C to do the work as B.

Let x=number of days it takes B.

Then $58x/15$=number of days it takes C.

$\therefore 5/x+30/58x=\frac{4}{9}$.

$\therefore\ x=12\frac{1}{2}\frac{2}{3}$ days, the time it takes B.

$58x/15=48$ days, the time it takes C.

Solved in like manner by *H. C. WHITAKER, P. S. BERG, B. F. SINE, J. F. TRAVIS,* and *P. H. PHILBRICK.*

NOTE.—This problem was proposed because we held that the wording of the problem was not sufficiently explicit. The fact that our contributors have taken the different views indicated by the above solutions confirms our opinion. The problem has been worded a little different in the last edition of Dr. Milne's Algebra. The wording of the problem is intended to lead to the first solution. B. F. F.

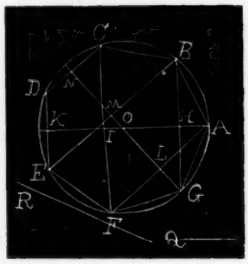
GEOMETRY.

122. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, Manchester High School, Manchester, N. H.

If perpendiculars are dropped from the vertices of a regular polygon upon any diameter of the circumscribed circle, the sum of the perpendiculars which fall on one side of this diameter is equal to the sum of those which fall on the opposite side. [From Chauvenet's *Treatise on Elementary Geometry.*]

I. Solution by HENRY HEATON, B. Sc., Atlantic, Ia.

Let $ABCDEFG$ represent a regular heptagon circumscribed by the circle whose center is O. Draw a diameter through A. It will be perpendicular to BG, CF, and DE, and it will bisect them in H, I, and K.



Draw any line RQ and for convenience call the perpendiculars from A, B, C, etc., upon RQ, a, b, c, etc. Then $b+g=2h$, $c+f=2i$, and $d+e=2k$.

Because K, I, H, and A are upon AK it can readily be shown that there is a point P upon AK such that $7p=2k+2i+2h+a=a+b+c+d+e+f+g$, and that this point is independent of the direction of RQ. In a similar manner it may be shown that there is a similar point p' upon the diameter through G, such that $7p'=a+b+c+d+e+f+g$. Hence $p=p'$. But this cannot be true without regard to the direction of RQ unless P and P' are at the intersection of the two diameters, that is, at the center. Hence the perpendicular from O upon any line RQ is one-seventh the sum of the perpendiculars a, b, c, d, e, f, and g. Hence if RQ passes through O, $a+b+c+d+e+f+g=0$.

It is evident that a similar demonstration may be made for any regular polygon.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and CHARLES C. CROSS, Whaleyville, Va.

In a polygon of an even number of sides a diameter drawn through one of the vertices will pass through another vertex which will be symmetrical with reference to the center of the circle as the center of symmetry. Consequently, the sum of the perpendiculars drawn from the vertices on one side of any diameter will be equal to the sum of the perpendiculars drawn from the vertices of the

other side of that diameter. If the number of sides of the polygon is uneven there will be no such symmetry.

Let MN be a diameter, $ABCDEFG$ be a regular polygon of n sides, n being an odd number, and O the center. Draw all the radii to the vertices, then $\angle ACB=\angle BOC=\angle COD=\ldots\ \ldots=2\pi/n=\beta$. Denote $\angle AOM$ by a, $\angle GOM$ by a'. It will be perceived at once that on one side of the diameter MN there are $\frac{1}{2}(n-3)$ triangles, and on the other $\frac{1}{2}(n-1)$ triangles, not cut by the diameter.

Employing the well-known formula of the difference of two cosines, we have

$$\cos(a-\tfrac{1}{2}\beta)-\cos(a-\tfrac{1}{2}\beta)=2\sin a\sin\tfrac{1}{2}\beta,$$

$$\cos(a+\tfrac{1}{2}\beta)-\cos(a+\tfrac{3}{2}\beta)=2\sin(a+\beta)\sin\tfrac{1}{2}\beta,$$

$$\cos(a+\tfrac{3}{2}\beta)-\cos(a+\tfrac{5}{2}\beta)=2\sin(a+2\beta)\sin\tfrac{1}{2}\beta$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\cos[a+\tfrac{1}{2}(2m-1)\beta]-\cos[a+\tfrac{1}{2}(2m+1)\beta]=2\sin(a+m\beta)\sin\tfrac{1}{2}\beta.$$

Adding,

$$\frac{\sin(a+\frac{1}{2}m\beta)\sin\frac{1}{2}(m+1)\beta}{\sin\frac{1}{2}\beta}=\sin a+\sin(a+\beta)+\sin(a+2\beta)+\ldots\ldots+\sin(a+m\beta).$$

Putting the radius of the circle=1, this last formula gives us the sum of the perpendiculars drawn from the vertices upon MN on either side of the latter.

Putting $m=\frac{1}{2}(n-3)\beta$, this sum

$$=\frac{\cos(a_1-\frac{3}{4}\beta)\cos\frac{1}{4}\beta}{\sin\frac{1}{2}\beta}.$$

Putting $m=\frac{1}{2}(n-1)$ we get

$$\frac{\cos(a_1-\frac{1}{4}\beta)\cos\frac{1}{4}\beta}{\sin\frac{1}{2}\beta},$$

but $a_1=\beta-a$, therefore the latter expression is

$$\frac{\cos(\frac{3}{4}\beta-a)\cos\frac{1}{4}\beta}{\sin\frac{1}{2}\beta}.$$

Consequently we have equality for the two sums of the perpendiculars, which proves the theorem.

Also solved by *G. B. M. ZERR.*

123. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

A étant le point d' intersection des médianes d' un triangle ABC, démontrerque $AB^2+BC^2+CA^2=3(GA^2+GB^2+GB^2)$. [Ex. 84, Géométrie. No. 2, 1re Anne *L' Éducation Mathématique.*]

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; CHARLES C. CROSS, Whaleyville, Va.; and H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Take $BC=a$, $AC=b$, and $AB=c$, and put the medians equal, respectively, m_a, m_b, and m_c. Then

$$m_a=\tfrac{1}{2}\sqrt{[2(b^2+c^2)-a^2]},$$
$$m_b=\tfrac{1}{2}\sqrt{[2(a^2+c^2)-b^2]},$$
$$m_c=\tfrac{1}{2}\sqrt{[2(a^2+b^2)-c^2]}.$$

But the medians of a triangle intersect at a common point two-thirds of the distance from the vertex to the middle of the opposite side. Whence

$$GA^2=(\tfrac{2}{3}m_a)^2=\tfrac{2}{9}b^2+\tfrac{2}{9}c^2-\tfrac{1}{9}a^2,$$
$$GB^2=(\tfrac{2}{3}m_b)^2=\tfrac{2}{9}a^2+\tfrac{2}{9}c^2-\tfrac{1}{9}b^2,$$
$$GC^2=(\tfrac{2}{3}m_c)^2=\tfrac{2}{9}a^2+\tfrac{2}{9}b^2-\tfrac{1}{9}c^2.$$

$\therefore GA^2+GB^2+GC^2=\tfrac{1}{3}(a^2+b^2+c^2)$, and $3(GA^2+GB^2+GC^2)=a^2+b^2+c^2=BC^2+AC^2+AB^2$.

II. Solution by J. W. YOUNG, Fellow and Assistant, Ohio State University, Columbus, O.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the triangle, AM, BN, and CL the medians, and G the intersection of the medians. Then by a well-known theorem,

$$AC^2+CB^2=2CL^2+2.AL^2=2CL^2+\tfrac{1}{2}AB^2.$$

Similarly,

$$CB^2+BA^2=2BN^2+\tfrac{1}{2}AB^2,$$
$$BA^2+AC^2=2AM^2+\tfrac{1}{2}CB^2.$$

Adding and dividing by 2, we have

$$AC^2+CB^2+BA^2=CL^2+BN^2+AM^2+\tfrac{1}{4}(AB^2+AC^2+CB^2),$$
$$\text{or } AC^2+CB^2+BA^2+\tfrac{4}{3}(CL^2+BN^2+AM^2),$$

(since G divides medians in the ratio 2:1)

$$=\tfrac{4}{3}(\tfrac{9}{4}CG^2+\tfrac{9}{4}BG^2+\tfrac{9}{4}AG^2)=3(CG^2+BG^2+AG^2.$$

The same propositions can very easily be proven analytically.

Solved in a similar manner by *COOPER D. SCHMITT, G. B. M. ZERR, WALTER H. DRANE,* and *CHAS. C. CROSS.*

124. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Every conic that passes through all the foci of a conic is a rectangular hyperbola. [From Charlotte A. Scott's *Modern Analytical Geometry.*]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If $u=0$.....(1) be the general equation of the second degree, we have the well-known equations

$$Fx+Gy-H=Cxy.....(2),$$

$$2Gx-2Fy-A+B=C(x^2-y^2).....(3),$$

for the foci, A, B, etc., being, as usual, the various minors of

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Either (2) or (3) is the rectangular hyperbola.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

"If through each of the two imaginary points at infinity on any circle two tangents be drawn to the conic, these tangents will form a quadrilateral whose vertices will be the real and imaginary foci of the conic."

Both Puckle and Salmon solve this problem. Salmon's proof expanded is as follows :

Find the condition that $x-x'+(y-y')_1/(-1)=0$ should touch the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The condition that $mx+ny+p=0$ touches the same conic is

$$(bc-f^2)m^2+(ca-g^2)n^2+(ab-h^2)p^2+2(gh-af)pn+2(hf-bg)pm$$
$$+2(fg-ch)mn=0,$$
$$\text{or } Am^2+Bn^2+Cp^2+2Fpn+2Gpm+2Hmn=0.$$

Now $m=1$, $n=_1/-1$, $p=-x'-y'_1/-1$ or dropping the accents $p=-(x+y_1/-1)$.

$\therefore A-B+Cx^2-Cy^2+2Cxy_1/(-1)-2Fx_1/(-1)+2Fy-2Gx-2Gy_1/(-1)+2H_1/(-1)=0.$

$\therefore C(x^2-y^2)+2Fy-2Gx+A-B=0.....(1),$

and $2Cxy-2Fx-2Gy+2H=0$, or

$$Cxy-Fx-Gy+H=0 .....(2).$$

The intersections of (1) and (2), both of which are rectangular hyperbolas, determine the foci.

NOTE ON PROBLEM 118.

BY W. H. CARTER, VICE PRESIDENT CENTENARY COLLEGE, JACKSON, LA.

The solution of this problem published on page 241 of the October (1899) number is somewhat unsatisfactory as it involves the Differential Calculus.

I use the figure given. Pass a circle through the points D and C, and tangent to the line FH. Let E be the point of tangency. Then E is the required point. For at any other point of FH, as M, the $\angle CMD$ has a numerical measure equivalent to one-half of the difference between those of the arc CD and the other arc intercepted by its sides, while at E, the $\angle CED$ has the same measure as one-half the arc CD. The visual angle is therefore a maximum at E. The calculation of the distance FE is then easy.

NOTE.—Prof. J. W. Scroggs, of Eureka, Kas., sent in a neat and brief construction of problem 118, but it came too late for credit in the proper issue.

CALCULUS.

93. Proposed by JOHN R. JEFFREY, Student in Ohio State University, Columbus, Ohio.

Solve the following differential equation :

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}+y=2x, \text{ when } x<1.$$

I. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Let $x=\sin\theta$, then $d\theta/dx=\sec\theta$.

$$\frac{d\theta}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}=\sec\theta\frac{dy}{d\theta}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\sec\theta\frac{dy}{d\theta}\right) = \frac{d}{d\theta}\left(\sec\theta\frac{dy}{d\theta}\right)\frac{d\theta}{dx}=\sec^2\theta\tan\theta\frac{dy}{d\theta}+\sec^2\theta\frac{d^2y}{d\theta^2}.$$

Substituting, our equation becomes

$$\frac{d^2y}{d\theta^2}+y=2\sin\theta\ldots\ldots (1).$$

Consider $\frac{d^2y}{d\theta^2}+y=0$.

$y=\cos\theta$ is a particular solution. Now let $y=z\cos\theta\ldots\ldots(2)$.
Differentiating we get

$$\frac{d^2y}{d\theta^2}=\cos\theta\frac{d^2z}{d\theta^2}-2\sin\theta\frac{dz}{d\theta}-z\cos\theta.$$

Substitute this, and y from (2), in (1), and we get

$$\frac{d^2z}{d\theta^2}-2\tan\theta\frac{dz}{d\theta}=z\tan\theta \ldots\ldots (3).$$

Let $p=dz/d\theta$. Then (3) becomes

$$\frac{dp}{d\theta}-2\tan\theta.p=z\tan\theta \ldots\ldots(4).$$

Consider now $\frac{dp}{d\theta}-2\tan\theta.p=0$. Its solution is $p\cos^2\theta=c\ldots\ldots(5)$.

Differentiate this, regarding c as variable, and we get

$$\frac{dp}{d\theta}=2\tan\theta.p+\frac{1}{\cos^2\theta}\frac{dc}{d\theta}.$$

This in (4) gives $\frac{dc}{d\theta}=\sin 2\theta\ldots\ldots(6)$.

$\therefore c=-\frac{1}{2}\cos 2\theta+k_1$. Substitute this value of c in (5) and we get

$$\frac{dz}{d\theta}=-1+\tfrac{1}{2}(1+2k_1)\sec^2\theta.$$

$\therefore z=-\theta+\frac{1}{2}(1+2k)\tan\theta+k_2$. This value of z in (2) gives

$$y=-\theta\cos\theta+\tfrac{1}{2}(1+2k)\sin\theta+k_2\cos\theta.$$

Substitute for θ and we get

$$y=-\sin^{-1}x.\sqrt{(1-x^2)}+\tfrac{1}{2}x(1+2k)+k_2\sqrt{(1-x^2)},$$

or in a more general form

$$y=Ax+B\sqrt{(1-x^2)}-\sqrt{(1-x^2)}\sin^{-1}x,$$

which is the complete primitive.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $y=vx$, then the equation becomes

$$x(1-x^2)\frac{d^2v}{dx^2}+(2-3x^2)\frac{dv}{dx}=2x.$$

$\therefore x^2(1-x^2)^{\frac{1}{2}}\frac{dv}{dx}=\sin^{-1}x-x\sqrt{(1-x^2)}+A.$

$\therefore \frac{dv}{dx}=\frac{\sin^{-1}x}{x^2(1-x^2)^{\frac{1}{2}}}+\frac{A}{x^2(1-x^2)^{\frac{1}{2}}}-1/x.$

$\therefore\ vx=Bx-(A+\sin^{-1}x)\sqrt{(1-x^2)}.$

$\therefore\ y=Bx-(A+\sin^{-1}x)\sqrt{(1-x^2)}.$

III. Solution by M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Ind., and GEORGE LILLEY, Ph.D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

This equation is exact and the first integral is

$$(1-x^2)\frac{dy}{dx}+xy=x^2+c_1, \text{ or } \frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{x^2}{1-x^2}+\frac{c_1}{1-x^2}.$$

This is a linear equation in which

$$\int P dx=\int\frac{xdx}{1-x^2}=\log\frac{1}{\sqrt{(1-x^2)}}.$$

$$\therefore\frac{1}{\sqrt{(1-x^2)}}y=\int\frac{x^2dx}{(1-x^2)^{\frac{3}{2}}}+c_1\int\frac{dx}{(1-x^2)^{\frac{3}{2}}}. \quad \text{Whence } x<1.$$

$$\frac{1}{\sqrt{(1-x)^2}}y=\frac{x}{\sqrt{(1-x^2)}}-\sin^{-1}x+\frac{c_1x}{\sqrt{(1-x^2)}}+c_2.$$

or $y=cx+\sqrt{(1-x^2)}(c_2-\sin^{-1}x)$ when c is written for c_1+1.

This example is found on page 144, *Johnson's Differential Equations*.

IV. Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

First solve $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0.$

Putting $\frac{dy}{dx}=p$, $(1-x^2)\frac{dp}{dx}-px+y=0.$

Differentiating, $(1-x^2)\frac{d^2p}{dx^2}-3x\frac{dp}{dx}=0.$

Putting q for $\frac{dp}{dx}$, $(1-x^2)\frac{dq}{dx}-3xq=0.$

Then $q=A(1-x^2)^{\frac{3}{2}}$, $p=\frac{Ax}{\sqrt{(1-x^2)}}+B$, $y=-A(1-x^2)^{\frac{1}{2}}+Bx.$

The particular integral may be found by the method of variation of parameters. Considering A and B variable, and substituting in the given equation,

$$-(1-x^2)^{\frac{3}{2}}\frac{d^2A}{dx^2}+(1-x^2)x\frac{d^2B}{dx^2}+x(1-x^2)^{\frac{1}{2}}\frac{dA}{dx}-x^2\frac{dB}{dx}=2x.$$

We may assume

$$x(1-x^2)^{\frac{1}{2}}\frac{dA}{dx}-x^2\frac{dB}{dx}=0.$$

Differentiating this and eliminating the second derivatives,

$$x(1-x^2)^{\frac{1}{2}}\frac{dA}{dx}+(1-x^2)\frac{dB}{dx}=-2x.$$

Solving for $\frac{dA}{dx}$ and $\frac{dB}{dx}$,

$$\frac{dB}{dx}=-2x,\ \frac{dA}{dx}=\frac{-2x^2}{(1-x^2)^{\frac{1}{2}}}.$$

Then $B=-x^2$, $A=\sin^{-1}x+x\sqrt{(1-x^2)}$.

The complete solution is then

$$y=(\sin^{-1}x-a)\sqrt{(1-x^2)}+(b-1)x,$$

a and b being arbitrary.

94. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

Find the minimum isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis. [Ex. 16, page 166, *Rice and Johnson's Differential Calculus.*]

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The equation of the tangent to an ellipse in terms of the slope being

$$y=mx+\sqrt{(a^2m^2+b^2)},$$

the altitude of the triangle will be the intercept of the tangent on the y-axis plus the semi-minor axis or $b+\sqrt{(a^2m^2+b^2)}$.

The base of the triangle found by putting $y=-b$ in the equation of the tangent is

$$\frac{-b-\sqrt{(a^2m^2+b^2)}}{m}.$$

The function of which a minimum value is to be found is accordingly

$$\frac{[b+\sqrt{(a^2m^2+b^2)}]^2}{m}.$$

The derivative of this function equated to zero gives after reduction

$$a^4m^4-3a^2b^2m^2=0.$$

The value of m is $\pm b/a\sqrt{3}$, and $a^2m^2+b^2=4b^2$.

The intercept on the y-axis is therefore $2b$, and the altitude of the triangle is $3b$.

MECHANICS.

NOTE.—Prize Problem No. 86, is open for solution as only one solution has been received, and that one is faulty.

93. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

A small rope, which is passed over a smooth pully, has attached at one end a weight of twenty pounds, and at the other end hangs a monkey, also weighing twenty pounds. Is it possible for the monkey to climb to the pulley, and if so, what will happen to the weight?

Solution by the PROPOSER.

When a body is in motion, its weight is altered, becoming greater when the body moves upward, and less when the body moves downward, and this increase or decrease in weight is proportional to the velocity. By this principle, when the monkey starts upward with a velocity, $ds/dt = \frac{1}{20}.Ft$, where F is the force with which he raises himself, his weight is increased by an amount proportional to $\frac{1}{20}Ft$. This force reacts undiminished upon the weight (as the pulley is smooth) and as the mass of the weight is also 20 pounds it must also begin to ascend with the same velocity as the monkey. This in turn increases the weight of the weight until it equals that of the monkey, which counteracts the effect of the monkey's increase in weight, and hence we have the result that monkey and weight will ascend with equal velocities.

94. Proposed by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School, Chester, Pa.

In a parallelogram $ABCD$, $\angle D = \beta$, $AB = a$, $BC = b$, the principal moments of inertia at the centroid are $(\frac{1}{24}m)[a^2+b^2 \pm \sqrt{(a^4-b^4+2a^2b^2\cos 2\beta)}]$ and the principal axes at the same point make with the side CD an angle θ given by

$$\tan 2\theta = \frac{b^2 \sin 2\beta}{a^2+b^2\cos 2\beta}.$$

Solution by the PROPOSER.

Let $ABCD$ be the parallelogram, O the intersection of the diagonals AC, BD, and OX, OY be the axes. For integration we will transform to OX, OY'.

$$\tan 2\theta = \frac{2\Sigma mxy}{\Sigma mx^2 - \Sigma my^2}.$$

Σmx^2 = moment of inertia about OX

$$= \rho \sin\beta \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} (x+y\cos\beta)^2 dxdy = \tfrac{1}{12}\rho ab \sin\beta (a^2+b^2\cos^2\beta) = \tfrac{1}{12}m(a^2+b^2\cos^2\beta).$$

$$\Sigma my^2 = \rho \sin^3\beta \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} y^2 dxdy = \tfrac{1}{12}mb^2\sin^2\beta.$$

$$\Sigma mxy=\rho\sin^2\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}(x+y\cos\beta)y\,dx\,dy=\tfrac{1}{24}mb^2\sin2\beta.$$

$$\therefore\ \tan2\theta=\frac{b^2\sin2\beta}{a^2+b^2(\cos^2\beta-\sin^2\beta)}=\frac{b^2\sin2\beta}{a^2+b^2\cos2\beta}.$$

Let A, B be the principal moments.

$\therefore\ A\cos^2\theta+B\sin^2\theta=\frac{1}{12}m(a^2+b^2\cos^2\beta)$.....(1).

$A\sin^2\theta+B\cos^2\theta=\frac{1}{12}mb^2\sin^2\beta$.....(2).

(1)+(2) gives $A+B=\frac{1}{12}m(a^2+b^2)$.

(1)−(2) gives $A-B=\frac{1}{12}m(a^2+b^2\cos2\beta)\sec2\theta$

$$=\tfrac{1}{12}m\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}.$$

$$\therefore\ A=\tfrac{1}{24}m[a^2+b^2+\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}].$$

$$B=\tfrac{1}{24}m[a^2+b^2-\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}].$$

DIOPHANTINE ANALYSIS.

76. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

It is required to find four positive numbers such that if each be diminished by twice the cube of their sum the four remainders will be rational cubes.

Solution by the PROPOSER.

Let u, v, x, y be the numbers.

Then $u-2(u+v+x+y)^3=a^3/h^3(x+y+u+v)^3$, suppose.

$v-2(u+v+x+y)^3=b^3/h^3(x+y+u+v)^3$, suppose.

$x-2(u+v+x+y)^3=c^3/h^3(x+y+u+v)^3$, suppose.

$y-2(u+v+x+y)^3=d^3/h^3(x+y+u+v)^3$, suppose.

Adding we get

$$u+v+x+y-8(u+v+x+y)^3=\frac{a^3+b^3+c^3+d^3}{h^3}(u+v+x+y)^3.$$

Let $a^3+b^3+c^3+d^3=h^3$.

$\therefore\ u+v+x+y=9(u+v+x+y)^3$. $\therefore\ u+v+x+y=\frac{1}{3}$.

$$\therefore\ u=\frac{a^3+2h^3}{27h^3},\ v=\frac{b^3+2h^3}{27h^3},\ x=\frac{c^3+2h^3}{27h^3},\ y=\frac{d^3+2h^3}{27h^3}.$$

Let $a=1$, $b=5$, $c=7$, $d=12$, $h=13$.

$\therefore\ u=\frac{43955}{593119},\ v=\frac{45119}{593119},\ x=\frac{47337}{593119},\ y=\frac{61122}{593119}.$

Let $a=4$, $b=7$, $c=8$, $d=17$, $h=18$.

$\therefore\ u=\frac{11728}{157464},\ v=\frac{12007}{157464},\ x=\frac{12176}{157464},\ y=\frac{16577}{157464}.$

Other values can be found for u, v, x, y.

77. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find (1) three consecutive numbers whose sum is a cube, and (2) three consecutive numbers the sum of whose cubes is a cube.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1). Let $n-1$, n, and $n+1$ be any three consecutive numbers.

Then $(n-1)+n+(n+1)=3n=$a cube$=27m^3$.

Whence $n=9m^3$.

$\therefore\ 9m^3-1$, $9m^3$, and $9m^3+1$ are the general expressions for three consecutive numbers whose sum is a cube.

Take $m=1$; then $8+9+10=27=3^3$.

Take $m=2$; then $71+72+73=216=6^3$; etc.

(2). $(n-1)^3+n^3+(n+1)^3=3n^3+6n=$a cube$=27m^3$.

Whence $n^3+2n=9m^3$.

Put $m=an$; then $n^3+2n=9a^3n^3$.

Whence $n^2+2=9a^3n^2$; and $n^2=2/(9a^3-1)$.

To obtain n integral, a must be fractional.

Put $a=1/b$; then $n^2=2b^3/(9-b^3)$.

To avoid imaginary results, $b<2\frac{1}{12}$.

The only integral values that can be assigned to b are 1 and 2.

Take $b=1$; then $n=\frac{1}{2}$.

Whence $(-\frac{1}{2})^3+(\frac{1}{2})^3+(\frac{3}{2})^3=(\frac{3}{2})^3$.

Take $b=2$; then $n=4$.

Whence $3^3+4^3+5^3=6^3$.

This is the only set of three consecutive integers the sum of whose cubes is a cube.

Fractional values of b give fractional values for n.

When $b=0$, $n=0$.

Whence $(-1)^3+0^3+1^3=0^3$.

Also solved by *CHARLES C. CROSS, JOSIAH H. DRUMMOND, ALOIS F. KOVARIK, NELSON L. RORAY, J. SCHEFFER, ELMER SCHUYLER,* and *G. B. M. ZERR.*

AVERAGE AND PROBABILITY.

81. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

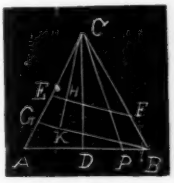
Find (1) the mean distance of all points on a side of an equilateral triangle from the opposite vertex; and (2), the average length of a line drawn at random across an equilateral triangle.

I. Solution by the PROPOSER.

1. Let each side$=a$ of the equilateral triangle ABC delineated in the annexed figure. Then its altitude $CD=\frac{1}{2}a\sqrt{3}$.

Put $DP=x$, then $CP=\sqrt{(x^2+\frac{3}{4}a^2)}$.

Hence the required mean distance is



$$M=\frac{\int_0^{\frac{1}{2}a}\sqrt{(x^2+\frac{3}{4}a^2)}\,dx}{\int_0^{\frac{1}{2}a}dx}=\tfrac{1}{4}a(2+3\log\sqrt{3}).$$

2. Suppose EF the random line. Draw GB parallel to EF, and CHK perpendicular to BG. Put $CH=x$ and $\angle CEF=\theta$. Then

$$EF=\frac{x\sin\frac{1}{3}\pi}{\sin\theta\sin(\frac{1}{3}\pi+\theta)}.$$

$CK=a\sin\theta$, and the required average length is

$$L=\frac{\int_0^{\frac{1}{3}\pi}\int_0^{a\sin\theta}x\sin\tfrac{1}{3}\pi\operatorname{cosec}\theta\operatorname{cosec}(\frac{1}{3}\pi+\theta)\,d\theta\,dx}{\int_0^{\frac{1}{3}\pi}\int_0^{a\sin\theta}d\theta\,dx}$$

$$=a\int_0^{\frac{1}{3}\pi}\sin\tfrac{1}{3}\pi\sin\theta\operatorname{cosec}(\tfrac{1}{3}\pi+\theta)\,d\theta=\tfrac{1}{4}a\int_0^{\frac{1}{3}\pi}[\sqrt{3}-3\cot(\tfrac{1}{3}\pi+\theta)]\,d\theta=\tfrac{1}{24}\pi a\sqrt{3}.$$

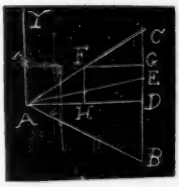
II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1). Let $AC=2a$, $DE=x$. Then $AD=a\sqrt{3}$, $AE=\sqrt{(3a^2+x^2)}$.

$$\therefore M=\frac{\int_0^a\sqrt{(3a^2+x^2)}\,dx}{\int_0^a dx}=\frac{1}{a}\int_0^a\sqrt{(3a^2+x^2}\,dx.$$

$\therefore M=\frac{1}{4}a(4+3\log 3)$.

If we regard $\angle DAE=\theta$ as the variable,



$$M=a\sqrt{3}\frac{\int_0^{\frac{1}{3}\pi}\sec\theta\,d\theta}{\int_0^{\frac{1}{3}\pi}d\theta}=\frac{6a\sqrt{3}}{\pi}\int_0^{\frac{1}{3}\pi}\sec\theta\,d\theta.$$

$$\therefore M=\frac{3a\sqrt{3}\log 3}{\pi}.$$

(2). Let FG be the random line, $AH=x$, $DG=y$.
Then $FH=\frac{1}{3}\sqrt{3}x$, $FG=\sqrt{[(a\sqrt{3}-x)^2+(y-\frac{1}{3}\sqrt{3}x)^2}=l$.

$$\triangle=\text{average length}=\frac{\int_0^{a\sqrt{3}}\int_0^a l\,dx\,dy}{\int_0^{a\sqrt{3}}\int_0^a dx\,dy}$$

$$\triangle=\frac{1}{a^2\sqrt{3}}\int_0^{a\sqrt{3}}\int_0^a \sqrt{[(a\sqrt{3}-x)^2+(y-\tfrac{1}{3}\sqrt{3}x)^2]}\,dx\,dy$$

$$=\frac{1}{12a^2\sqrt{3}}\int_0^{a\sqrt{3}}\Big[4(a\sqrt{3}-x)^2+x\sqrt{[9a^2+(4x-3\sqrt{3}a)^2]}$$

$$-6(a\sqrt{3}-x)^2\log\Big(\frac{\sqrt{[9a^2+(4x-3\sqrt{3}a)^2]}-2x}{6(\sqrt{3}-x)}\Big)\Big]dx.$$

$$\therefore\ \triangle=(a/16)[6+2\sqrt{3}+4\log 3+\log(3+2\sqrt{3})].$$

82. Proposed by **B. F. FINKEL, A. M., M. Sc.**, Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average area of the quadrilateral formed by joining the extremities of two chords perpendicular to each other and passing through a point at a distance a from the center of a circle radius R.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School, Chester, Pa.; and the **PROPOSER**.

Let $\triangle$=average area, $\angle BPC=\theta$, P the given point, C the center of the cirle, $CP=a$, $CB=R$.

CASE I. $a<R$, $CG=a\sin\theta$, $CF=a\cos\theta$.

$\therefore\ AB=2\sqrt{(R^2-a^2\sin^2\theta)}$, $DE=2\sqrt{(R^2-a^2\cos^2\theta)}$.

Area $ADBE=2\sqrt{[(R^2-a^2\sin^2\theta)(R^2-a^2\cos^2\theta)]}$
$=2R^2\sqrt{[(1-e^2\sin^2\theta)(1-e^2\cos^2\theta)]}$ where $e^2=a^2/R^2$.

$$\therefore\ \triangle=2R^2\int_0^{\frac{1}{2}\pi}\sqrt{[(1-e^2\sin^2\theta)(1-e^2\cos^2\theta)]}\,d\theta\Big/\int_0^{\frac{1}{2}\pi}d\theta$$

$$=\frac{2R^2}{\pi}(2-e^2)\,E\Big(\frac{e^2}{2-e^2},\ \tfrac{1}{2}\pi\Big).$$

(See page 356, No. 10, Vol. I, for above integration.

CASE II. $a>R$ and $a<R\sqrt{2}$, $PF=a\cos\theta$, $PG=a\sin\theta$.
$PB=a\cos\theta+\sqrt{(R^2-a^2\sin^2\theta)}$, $PA=a\cos\theta-\sqrt{(R^2-a^2\sin^2\theta)}$.
$PE=a\sin\theta+\sqrt{(R^2-a^2\sin^2\theta)}$, $PD=a\sin\theta-\sqrt{(R^2-a^2\cos^2\theta)}$.
Area $ABED=\frac{1}{2}(BP.PE-AP.PD)$
$=a[\sqrt{(R^2-a^2\sin^2\theta)}\sin\theta+\sqrt{(R^2-a^2\cos^2\theta)}\cos\theta]$.
The limits of θ are $\frac{1}{2}\pi-\sin^{-1}(R/a)=\theta''$ and $\sin^{-1}(R/a)=\theta'$.

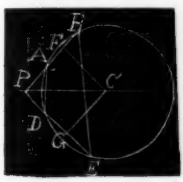
$$\therefore\ \Delta = aR\int_{\theta'}^{\theta}[\sqrt{(1-e^2\sin^2\theta)}\sin\theta+\sqrt{(1-e^2\cos^2\theta)}\cos\theta]d\theta/\int_{\theta'}^{\theta}d\theta$$

$$=\frac{aR}{4\sin^{-1}(R/a)-\pi}\Big[\sin\theta'\sqrt{(1-e^2\cos^2\theta')}$$

$$-\sin\theta''\sqrt{(1-e^2\cos^2\theta'')}-\cos\theta'\sqrt{(1-e^2\sin^2\theta')}$$

$$+\cos\theta''\sqrt{(1-e^2\sin^2\theta'')}$$

$$+\frac{1-e^2}{e}\log\left(\frac{e\sin\theta'+\sqrt{(1-e^2\cos^2\theta')}}{e\sin\theta''+\sqrt{(1-e^2\cos^2\theta'')}}\right)$$



$$-\frac{1-e^2}{e}\log\left(\frac{e\cos\theta'+\sqrt{(1-e^2\sin^2\theta')}}{e\cos\theta''+\sqrt{(1-e^2\sin^2\theta'')}}\right)\Big].$$

But $\sin\theta'=\cos\theta''=R/a$, $\sin\theta''=\cos\theta'=[\sqrt{(a^2-R^2)}]/a$.
$e^2\sin^2\theta'=e^2\cos^2\theta''=1$, $e^2\sin^2\theta''=e^2\cos^2\theta'=[(a^2-R^2)/R^2]=e^2-1$.
Whence by substitution and reduction we get

$$\Delta=\frac{R^2}{2\sin^{-1}(R/a)-\frac{1}{2}\pi}\Big[\sqrt{(2-e^2)}+(1-e^2)\log\Big(\frac{1+\sqrt{(2-e^2)}}{\sqrt{(e^2-1)}}\Big)\Big].$$

83. Proposed by **F. P. MATZ**, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the average area of all ellipses whose semi-axis major is a.

I. Solution by **J. W. YOUNG**, Fellow and Assistant, Ohio State University, Columbus, Ohio.

The area of an ellipse whose major-axis is a, and whose minor-axis is b, is πab. We must find the average of all possible values of this expression as b varies from zero to a.

$$\therefore\ \text{Average required}=\frac{\pi a\int_0^a b\,db}{\int_0^a db}=\tfrac{1}{2}\pi a^2,$$

$=\frac{1}{2}$ the area of the circle whose radius is the major-axis of the ellipse.

II. Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

1. Let x=semi-conjugate axis. Then average area

$$=\pi a\frac{\int_0^a x\,dx}{\int_0^a dx}=\tfrac{1}{2}\pi a^2=1.5708a^2.$$

2. Let e=eccentricity. Then area$=\pi a^2\sqrt{(1-e^2)}$.

$$\therefore \text{ Average area}=\pi a^2\,\frac{\int_0^1\sqrt{(1-e^2)}}{\int_0^1 de}=\tfrac{1}{4}\pi^2a^2=2.4674a^2.$$

MISCELLANEOUS.

73. Proposed by CHAS. E. MYERS, Canton, Ohio.

In an ice cream freezer, cream of a homogeneous character and at the uniform temperature of 60° Fahrenheit is put into a cylinder having a closed base, and the whole put into a freezing mixture so as to subject the base and convex surface to a constant temperature of 30° Fahrenheit. Required the temperature at any point within the cream after the expiration of a given time. [From *Higher Mathematics.*]

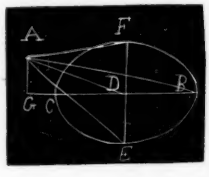
No solution of this problem has been received.

74. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

The longest diameter of a horizontal ellipse is $CB=2a=6$ feet. Its shortest diameter is $EF=2b=4$ feet, their intersection being at D. Find in an indefinite vertical plane passing through CB, a point $A=5$ feet$=c$ from D, the ellipse being seen from A as a circle.

I. Solution by the late B. F. BURLESON, and the PROPOSER.

The eye being at A, and the ellipse being projected as a circle, CB and EF subtend equal angles at A, or $\angle EAF=\angle BAC$: Produce DC to G, A being vertically over G, and put $CG=x$, and $GA=y$, and $\angle ADC=\phi$=angle of elevation of A.



Then $y=\sqrt{[c^2-(a+x)^2]}$ .....(1).

$$AB=\sqrt{[(2a+x)^2+y^2]} \quad .....(\alpha).$$

$\sin\angle ACG=\sin\angle ACB=y/\sqrt{(x^2+y^2)}$.....($\beta$). and $\tan\angle EAD=b/c$ ....(γ).

$\therefore \sin\angle EAF=\sin\angle BAC=2dc/(b^2+c^2)$..... ($\delta$).

From $\triangle BAC$ we have the proportion, $AB:\sin\angle ACB::BC:\sin\angle BAC$.

$$\therefore \frac{2bc\sqrt{[(2a+x)^2+y^2]}}{b^2+c^2}=\frac{2ay}{\sqrt{(x^2+y^2)}}\text{.....(2).}$$

Resolving (1) and (2) we have $x=\dfrac{c\sqrt{[(c^4-a^2b^2)(a^2-b^2)]}-a}{a(c^2-b^2)}$

$$=\tfrac{5}{6}\tfrac{3}{1}\sqrt{(2945)}-3=1.30697255 \text{ feet.}$$

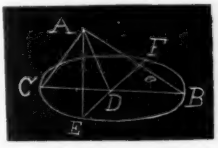
$$\therefore\ y=\frac{bc(c^2-a^2)}{a(c^2-b^2)}=2\tfrac{3}{6}\tfrac{4}{3} \text{ feet.}$$

$$\phi = \sin^{-1}\left(\frac{b(c^2-a^2)}{a(c^2-b^2)}\right) = \sin^{-1}(\tfrac{3}{6}\tfrac{2}{3}) = 30^\circ\ 31'\ 35\tfrac{1}{4}''.$$

II. Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, Ohio.

If the ellipse is seen from A as a circle, $\angle EAF = \angle CAB$. This relation enables us to calculate the coördinates of the point A, in the vertical plane through CB, the origin being at D, and CB being the axis of x.

$$\tan\tfrac{1}{2}\angle EAF = b/c.$$



Also let $AC = q$ and $AB = p$; $CB = 2a$.

Then $\tan\frac{1}{2}\angle CAB = \sqrt{\left(\frac{(s-p)(s-q)}{s(s-2a)}\right)}$

where $s = \frac{1}{2}(p+q+2a)$.

Since $\angle EAF = \angle CAB$, we have, after reducing,

$$\frac{b^2}{c^2} = \frac{4a^2-(p-q)^2}{(p+q)^2-4a^2} \quad \ldots\ldots (1).$$

Now, since $AD = c$ is the median of $\triangle ABC$, we have by a common trigonometrical formula,

$$2c^2 = p^2+q^2-2a^2 \text{ or } p^2+q^2 = 2(a^2+c^2) \quad \ldots\ldots (2).$$

From equations (1) and (2), we obtain

$$p+q = \pm 2\sqrt{\left(\frac{a^2b^2-c^4}{b^2-c^2}\right)};\ p-q = \pm 2c\sqrt{\left(\frac{b^2-a^2}{b^2-c^2}\right)}.$$

Now, to obtain x and y, we have the relations,

$$(a+x)^2+y^2 = p^2,\ (a-x^2)+y^2 = q^2,$$

from which $x = \dfrac{p^2-q^2}{4a}$.

Substituting from above, we have

$$x = \frac{\pm c}{a(b^2-c^2)}\sqrt{[(a^2b^2-c^4)(b^2-a^2)]}, \quad y = \pm\frac{bc(a^2-c^2)}{a(b^2-c^2)}.$$

In the special case given, where $a=3$, $b=2$, $c=5$, we have

$$x = \pm 4.31 \text{ feet}, \quad y = \pm 2.54 \text{ feet}.$$

NOTE.—Other solutions of this problem will appear next month.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

126. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Bought 150 head of stock for \$300, paying for each kind $\$2\frac{5}{6}$, $\$1\frac{5}{6}$, and $\$\frac{5}{7}$, respectively. Find number of each kind bought.

127. Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

A man borrows \$1000 of a Building and Loan Association, and at the same time subscribes for 10 \$100-shares of stock. A membership fee of \$1 per share is charged. At the beginning of each month an installment of \$1 per share is paid, also 5% interest and 5% premium on the \$1000. The stock matures in 75 months and the debt is cancelled. What rate of interest does he pay per annum?

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

GEOMETRY.

138. Proposed by JOHN M. HOWIE, Professor of Mathematics, The Nebraska State Normal, Peru, Neb.

K is the middle point of any chord AB of a given circle. CD and EF are any two chords passing through K. CF and ED intersect AB at M and N, respectively. Prove that KM equals KN.

139. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If $x^2+y^2=1$ [x and y being points corresponding to complex numbers], prove that x and y are at the ends of conjugate radii of an ellipse whose foci are ± 1. [From *Harkness and Morley's Introduction to the Theory of Functions.*]

140. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.

Having given two points on a range and a point that bisects the distance between two other points that form an harmonic ratio with the given points, give, if possible, a geometrical construction for locating the other two points.

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

MECHANICS.

103. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Given the lengths a, b of the sides of a parallelogram, the direction of side a, and the position of the centroid. Prove that the locus of the foci of the ellipse of gyration at the centroid is a Cassinian Oval, having its foci distant $a/2\sqrt{3}$ from the centroid, and the constant product of its focal distances equal to $\frac{1}{12}b^2$.

104. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

From a locomotive and tender standing still on a bridge, the pressure on the bridge is $p_1=80$ tons. The track is supposed to be straight and practically

horizontal. Had the locomotive and tender been running at the rate of $r=60$ miles an hour, how many tons would the pressure on the bridge have been?

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

AVERAGE AND PROBABILITY.

91. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Six points A, B, C, D, E, F are taken at random on the surface of a sphere. Find the chance that the plane through A, B, C intersects the plane through D, E, F within the sphere.

92. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A circular field, radius r, is divided into four *equal* parts, by concentric circles and three concentric rings. From the center of this field are fired *at random*, and with such a velocity as not to produce a range greater than the radius of the field, $m=1000$ projectiles of the *same* kind. How many projectiles should have fallen into each one of these four equal parts of the field?

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

EDITORIALS.

Mr. W. D. Cairns has been appointed to an instructorship in mathematics in Oberlin College.

A fac-simile reprint of Legendre's Theorie des Numbers has lately been issued by A. Hermann, of Paris.

Among the lecturers before the students of the College for Women of the Western Reserve University, is Prof. E. S. Loomis, Teacher of Mathematics in the West High School of Cleveland, Ohio. Prof. Loomis's lectures are on Fundamentals in the Teaching of Arithmetic, and Essentials in Teaching Algebra.

BOOKS AND PERIODICALS.

Elements of Precise Surveying and Geodesy. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. 8vo. Cloth, 261 pages. Price, $2.50. New York: John Wiley & Sons.

The work begins with an elementary treatment of the method of least squares, developing the theory of the method in such an elegant and lucid way as to be clearly comprehended by a beginner. Many examples are solved to illustrate the various principles as they are developed. One not familiar with the Law of Probability of Error, and the Method of Least Squares and desiring to get a working knowledge of the subject needs this book. Chapter II treats of Precise Plane Triangulation in which is applied the Method of Least Squares for the correction of measured magnitudes; Chapter III treats of Base

Lines; Chapter IV, of Leveling; Chapter V, of Astronomical Work, in which is treated the Precise Measurement of Latitude by various methods; Longitude, Azimuth, and Time; Chapter VI treats of Spherical Geodesy. This subject is introduced by a brief but very interesting history of Geodesy, beginning with the earliest times and bringing it down to the present time. Chapter VII deals with Spheriodal Geodesy; Chapter VIII, Geodetic Coördinates and Projections, including the various map-projections; Chapter IX, of Geodetic Triangle; and Chapter X, on the Figure of the Earth. This last chapter is full of interest, as the various assumed forms of the earth are discussed briefly. These various forms are the Spheriod, the Ellipsoid, the Ovaloid, and the Geoid. The book is one of the highest interest and importance not only to engineers but to mathematicians as well.

B. F. F.

Statistical Methods with Special Reference to Biological Variation. By C. B. Davenport, Instructor in Zoölogy in Harvard University. 16mo. Morocco, 135 pages. Price, $1.25. New York: John Wiley & Sons.

This work is intended especially for Botanists, Zoölogists, Anthropologists, Anatomists, Physiologists, and Psychologists who are interests in the quantitative study of species and of organic variation. It will also be of service to Economists, Sociologists, Meteorologists, and practical Statisticians. It treats in simple language, and for the most part without the use of mathematics beyond the elements of algebra of the statistical methods elaborated by Galton and Pearson. The mean, mode, probable error, index of variation, the coefficients of correllation and heredity are defined and the methods of getting them explained. The treatment of curves of the different classes, normal, skew (of three types), compound, and multimodal, is made clear. The work is a complete handbook, for it contains tables of reduction from English to metric units; squares, cubes, roots, and reciprocals of numbers from 1 to 1000; six-place logarithms of numbers and circular functions; table of gamma functions, etc. The table of using each table is fully explained. There have been added also some pages of cross-section paper, a metric scale, and a protractor. The book contains 31 figures, is of pocket size, and is bound in morocco. B. F. F.

Transactions of the American Mathematical Society, Edited by Drs. E. H. Moore, E. W. Brown, and T. S. Fiske. Published quarterly by the Society with the coöperation of Harvard, Yale, Princeton, Columbia, Northwestern, Cornell, The University of Chicago, Haverford College, Bryn Mawr College, and the University of Califonia.

The first number of the first volume contains the following articles: Conics and Cubics connected with a Plane Cubic by certain Covariant Relations, by H. S. White; Formenthoretische Entwickelung der Herrn White's Abhandlung über curven dritter Ordung enthallenten Sätze, by P. Gordan; Sur la définition générale des functions analytiques, d' apès Cauchy, by E. Goursat; On a Class of Particular Solutions of the Problem of Four Bodies, by F. R. Moulton; Definition of the Abelian, the Two Hypoabelian, and the Related Linear Groups as Quotient-groups of the Groups of Isomorphisms of Certain Elementary Groups, by L. E. Dickson; Note on the Unilateral Surface of Moebius, by H. Maschke; On Regular Singular Points of Linear Differential Equations of the Second Order whose Coefficients are not necessarily Analytic, by M. Bocher; The Elliptic Sigma-Functions considered as a Special Case of the Hyperelliptic Sigma-Functions, by O. Bolza; On Groups which are the Direct Products of Two Subgroups, by G. A. Miller; On Certain Crinckly Curves, by E. H. Moore; A Definition of the General Abelian Linear Group, by L. E. Dickson.

B. F. F.

Annals of Mathematics. Published under the Auspices of Harvard University. Issued Quarterly. Price, $2.00 per year in advance.

The January (1900) number contains the following: On Three-Dimensional Determinants, by E. R. Hedrick; On Tide Currents in Estuaries and Rivers, by Ernest W. Brown; Note on Netto's Theory of Substitution, by Dr. G. A. Miller; A Method of Solving Determinants, by G. Macloski; The Development of Functions, by S. A. Corey; Illustrations of the Elliptic Integrals of the First Kind by a certain Link-Work, Dr. Arnold Emch; and Problems in the Theory of Continuous Groups, by Chas. L. Bouton.

B. F. F.

Periodico di Mathematica for January and February, contains among other articles, one on the Geometric Theory of Inversion, by G. Lazzeri, and another on the Fundamental Theorem of Maxima and Minima, by Roberto Volpi.

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ON THE TRANSCENDENTAL FORM OF THE RESULTANT.

By DR. E. D. ROE, Jr., Elmira, N. Y.

(A paper presented December 28, 1899, at the meeting of the Chicago Section of the American Mathematical Society.)*

I. DERIVATION AND PROPERTIES.

§1. INTRODUCTION.

The writer was incited to the following investigation by portions of letters written by Professor Gordan to M. Hermite and himself. Under date of October 17, 1898, in a letter to the writer Professor Gordan says: "Ich habe die Untersuchung der Resultante weiter verfolgt und die Formel gefunden, sie durch Potenzsummen s der beiden Gleichungen auszudruecken. Ich habe sie nach Paris gesandt, wo sie demnächst in den Comptes Rendus erscheinen wird. R sei die Resultante der Gleichungen

$$f=x^m+a_1x^{m-1}+a_2x^{m-2}+\ldots\ldots$$

$$1+\phi=1+b_1x+b_2x^2+\ldots\ldots$$

Die $s_\rho(a)$ sind die Potenzsummen der Wurzeln α der Gleichung f und die $s_\rho(b)$

*The writer wishes here to acknowledge his obligations to Prof. H. S. White for presenting this paper and for subsequent criticisms for the revision. He wishes also to thank Prof. M. Bocher for friendly criticism.

dieselben Functionen der b. Die Producte der s bezeiche ich durch S: $S_\lambda = (s_{\lambda_1})^{\rho_1}(s_{\lambda_2})^{\rho_2}\ldots$ und fuehre die Zahl g_λ ein, $g_\lambda = (\lambda_1)^{\rho_1}(\lambda_2)^{\rho_2}\ldots\ldots\rho_1!\,\rho_2!\ldots\ldots$ Es ist dann:

$$R = \Sigma \frac{(-1)^{\rho_1+\rho_2+\ldots\ldots}}{g_\lambda} S_\lambda(a)\, S_\lambda(b).$$

Die Summe erstreckt sich ueber alle λ, ρ, fuer welche

$$\lambda_1\rho_1 + \lambda_2\rho_2 + \ldots\ldots \leq mn$$

ist.

Aus diesser Formel kann man leicht die Entwickelung der symmetrischen Functionen

$$\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\alpha_3^{\kappa_3}\ldots\ldots = \beta_{\kappa_1}\beta_{\kappa_2}\beta_{\kappa_3}\ldots\ldots$$

sowohl als ganze Function der s als auch als ganz Function der Coefficienten a ableiten. In der Formel

$$\beta_{\kappa_1}\beta_{\kappa_2}\ldots\ldots\beta_{\kappa_\nu} = \Sigma(-1)^{\nu+\rho_1+\rho_2+\ldots\ldots} C_{\lambda,\kappa} S_\lambda$$

sind die Coefficienten $C_{\lambda,\kappa}$ mittelst der Formel gegeben:

$$C_{\lambda,\kappa} = \Sigma \frac{1}{\tau_1!\tau_2!\ldots\ldots}(\varkappa_{11}\varkappa_{12}\ldots\ldots\varkappa_{1\nu_1} \mid \varkappa_{21}\varkappa_{22}\ldots\ldots\varkappa_{2\nu_2} \mid \varkappa_{31}\varkappa_{32}\ldots\ldots\varkappa_{3\nu_3} \mid \ldots\ldots).$$

In ihr bedeutet der Klammerausdruck das Product $(\nu_1-1)!(\nu_2-1)!(\nu_3-1)!$ .... und die τ sowohl die Anzahl gleicher Zahlen in Combinationen $\varkappa_{\mu_1}\varkappa_{\mu_2}\ldots\ldots$ als auch die Anzahl von Gleichen Combinationen. Die Summe erstreckt sich ueber alle Combinationen der Zahlen $\varkappa_1\varkappa_2\ldots\ldots\varkappa_\nu$ bei denen die $\varkappa_{\mu_1}+\varkappa_{\mu_2}-\varkappa_{\mu_3}\ldots\ldots = \lambda_\mu$ sind. In der Formel

$$\beta_{\kappa_1}\beta_{\kappa_2}\ldots\ldots\beta_{\kappa_\nu} = \Sigma(-1)^{\nu+\nu_1+\rho_1+\rho_2+\ldots\ldots}\varkappa_1\varkappa_2\ldots\ldots \mid \mu_1\mu_2\ldots\ldots a_{\mu_1}a_{\mu_2}\ldots\ldots a_{\mu_{\nu_1}}$$

lassen sich die Coefficienten $\varkappa_1\varkappa_2\ldots\ldots \mid \mu_1\mu_2\ldots\ldots$ mittelst der Formel berechnen

$$\varkappa_1\varkappa_2\ldots\ldots \mid \mu_1\mu_2\ldots\ldots = \Sigma g_\lambda \frac{1}{\tau_1!\,\tau_2!}(\varkappa_{11}\varkappa_{12}\ldots\ldots \mid \varkappa_{21}\varkappa_{22}\ldots\ldots \mid \ldots\ldots)$$

$$(\mu_{11}\mu_{12}\ldots\ldots \mid \mu_{21}\mu_{22}\ldots\ldots \mid \ldots\ldots)$$

wo die $\varkappa_{\sigma_1}\varkappa_{\sigma_2}\ldots\ldots\mu_{\sigma_1}\mu_{\sigma_2}\ldots\ldots$ Combinationen der $\varkappa$ und μ darstellen, welche durch die Relationen ver knuepft sind $\varkappa_{\sigma_1}+\varkappa_{\sigma_2}+\ldots\ldots = \mu_{\sigma_1}+\mu_{\sigma_2}+\ldots\ldots = \lambda_\sigma$.

Sie sehen in beiden Fällen geschieht die Berechnung durch independte Formeln und nicht durch Recursionen. Mit einiger Uebung genuegen bis zum Grade 8 diese Formeln zur Berechnung eines Coefficienten ca 3 Minuten, Recursionsformeln sind nicht mehr noethig."